

EXPONENTIAL SUMS ON \mathbf{A}^n , II

ALAN ADOLPHSON AND STEVEN SPERBER

ABSTRACT. We prove a vanishing theorem for the p -adic cohomology of exponential sums on \mathbf{A}^n . In particular, we obtain new classes of exponential sums on \mathbf{A}^n that have a single nonvanishing p -adic cohomology group. The dimension of this cohomology group equals a sum of Milnor numbers.

1. INTRODUCTION

Let p be a prime number, $q = p^a$, and \mathbf{F}_q the finite field of q elements. Associated to a polynomial $f \in \mathbf{F}_q[x_1, \dots, x_n]$ and a nontrivial additive character $\Psi : \mathbf{F}_q \rightarrow \mathbf{C}^\times$ are exponential sums

$$(1.1) \quad S(\mathbf{A}^n(\mathbf{F}_{q^i}), f) = \sum_{x_1, \dots, x_n \in \mathbf{F}_{q^i}} \Psi(\text{Trace}_{\mathbf{F}_{q^i}/\mathbf{F}_q} f(x_1, \dots, x_n))$$

and an L -function

$$(1.2) \quad L(\mathbf{A}^n, f; t) = \exp\left(\sum_{i=1}^{\infty} S(\mathbf{A}^n(\mathbf{F}_{q^i}), f) \frac{t^i}{i}\right).$$

One of the basic results on exponential sums is the following theorem of Deligne[3, Théorème 8.4]. Let $\delta = \deg f$ and write

$$(1.3) \quad f = f^{(\delta)} + f^{(\delta-1)} + \dots + f^{(0)},$$

where $f^{(j)}$ is homogeneous of degree j .

Theorem 1.4. *Suppose $(p, \delta) = 1$ and $f^{(\delta)} = 0$ defines a smooth hypersurface in \mathbf{P}^{n-1} . Then $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree $(\delta - 1)^n$, all of whose reciprocal roots have absolute value $q^{n/2}$.*

For exponential sums on \mathbf{A}^n , several generalizations of Deligne's result have been proved ([1, 2, 5, 7]). In all these theorems, the hypothesis implies that f , regarded as a function from \mathbf{A}^n to \mathbf{A}^1 , has only finitely many critical points and the degree of the polynomial $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ equals the sum of the Milnor numbers of those critical points. We examine these critical points more closely.

We write $\mathbf{F}_q[x]$ for $\mathbf{F}_q[x_1, \dots, x_n]$ and consider the complex $(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^*, \phi_f)$, where $\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k$ denotes the module of differential k -forms of $\mathbf{F}_q[x_1, \dots, x_n]$ over \mathbf{F}_q and $\phi_f : \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k \rightarrow \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k+1}$ is defined by

$$\phi_f(\omega) = df \wedge \omega,$$

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where $d : \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k \rightarrow \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k+1}$ is the exterior derivative. The map $f : \mathbf{A}^n \rightarrow \mathbf{A}^1$ has only isolated critical points if and only if

$$(1.5) \quad H^i(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_f) = 0 \quad \text{for } i \neq n,$$

which implies that $\dim_{\mathbf{F}_q} H^n(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_f)$ is finite. Since

$$H^n(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_f) \simeq \mathbf{F}_q[x_1, \dots, x_n]/(\partial f/\partial x_1, \dots, \partial f/\partial x_n),$$

we have

$$\dim_{\mathbf{F}_q} H^n(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_f) = M_f,$$

where M_f denotes the sum of the Milnor numbers of the critical points of f .

We consider one approach to verifying (1.5). Every $\omega \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k$ can be uniquely written in the form

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} \omega(i_1, \dots, i_k) dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

with $\omega(i_1, \dots, i_k) \in \mathbf{F}_q[x]$. If each coefficient $\omega(i_1, \dots, i_k)$ is a homogeneous form of degree l , we call ω *homogeneous* and write

$$\begin{aligned} \deg_{\text{coeff}} \omega &= l \\ \deg \omega &= l + (n - k)(\delta - 1). \end{aligned}$$

The point of the latter definition is that we can define an increasing filtration F on $\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k$ by setting

$$F_l \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k = \text{the } \mathbf{F}_q\text{-span of homogeneous } k\text{-forms } \omega \text{ with } \deg \omega \leq l,$$

and $(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_f)$ then becomes a filtered complex. Consider the associated spectral sequence

$$(1.6) \quad E_1^{r,s} = H^{r+s}(F_r/F_{r-1}(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_f)) \Rightarrow H^{r+s}(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_f).$$

The E_1 -terms are just the cohomology of the homogeneous pieces of the associated graded complex to $(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_f)$, which may be identified with $(\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^\bullet, \phi_{f^{(\delta)}})$. Since this latter complex is isomorphic to the Koszul complex on $\mathbf{F}_q[x_1, \dots, x_n]$ defined by $\{\partial f^{(\delta)}/\partial x_i\}_{i=1}^n$, the assertion that

$$(1.7) \quad E_1^{r,s} = 0 \quad \text{for } r + s \neq n$$

is equivalent to the assertion that

$$(1.8) \quad \{\partial f^{(\delta)}/\partial x_i\}_{i=1}^n \text{ form a regular sequence in } \mathbf{F}_q[x_1, \dots, x_n].$$

It follows from (1.8) that

$$\dim_{\mathbf{F}_q} \mathbf{F}_q[x_1, \dots, x_n]/(\partial f^{(\delta)}/\partial x_1, \dots, \partial f^{(\delta)}/\partial x_n) = (\delta - 1)^n,$$

hence

$$(1.9) \quad \dim_{\mathbf{F}_q} \bigoplus_{r+s=n} E_1^{r,s} = (\delta - 1)^n.$$

The spectral sequence (1.6) and condition (1.7) imply that (1.5) holds, and (1.9) then implies that $M_f = (\delta - 1)^n$. Theorem 1.4 of [2], a slight generalization of Theorem 1.4 above, can then be reformulated as follows.

Theorem 1.10. *Suppose (1.7) holds. Then $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree M_f , all of whose reciprocal roots have absolute value $q^{n/2}$.*

Condition (1.5) holds if there exists a positive integer e such that

$$(1.11) \quad E_e^{r,s} = 0 \quad \text{for } r + s \neq n.$$

We are interested in determining the extent to which the conclusion of Theorem 1.10 holds when condition (1.7) is replaced by condition (1.11) for some $e > 1$. In general, some additional hypothesis is needed, as is illustrated by the one-variable example $f(x_1) = x_1^p - x_1$ over the field \mathbf{F}_p . The purpose of this paper is to prove a result that provides evidence for such a theorem.

Dwork has associated to f a complex $(\Omega_{C(b)}, D)$ (of length n) depending on a choice of rational parameter b satisfying $0 < b < p/(p-1)$ (we review this theory in section 2). Each $\Omega_{C(b)}^i$, $i = 0, \dots, n$, is a p -adic Banach space over a field $\tilde{\Omega}_0$ (a finite extension of \mathbf{Q}_p) and is equipped with a Frobenius operator α_i commuting with the differential D of the complex. Furthermore,

$$(1.12) \quad L(\mathbf{A}^n, f; t) = \prod_{i=0}^n \det(I - t\alpha_i \mid H^i(\Omega_{C(b)}, D))^{(-1)^{i+1}}.$$

Theorem 1.13. *Suppose there exist e, m such that $E_e^{r,s} = 0$ for all r, s such that $r + s = m$. Then for*

$$(1.14) \quad \frac{\delta}{(p-1)(\delta-e+1)} < b < \frac{p\delta}{(p-1)\delta+e-1}$$

we have

$$H^m(\Omega_{C(b)}, D) = 0.$$

If (1.11) holds, then, in addition, for b in the range (1.14) we have

$$\dim_{\tilde{\Omega}_0} H^n(\Omega_{C(b)}, D) = M_f.$$

Remark. It is easily seen that in (1.14) the upper bound for b is greater than the lower bound for b if and only if

$$(1.15) \quad \left(1 + \frac{p}{(p-1)^2}\right)(e-1) < \delta,$$

i. e., (1.15) is equivalent to the existence of rational b satisfying (1.14). For example, if $e = 2$, then this condition requires $\delta \geq 2$ for odd primes p and $\delta \geq 4$ for $p = 2$. In general, for p sufficiently large relative to δ , it becomes simply $e \leq \delta$.

By [11, section 3.4], α_n is invertible on $H^n(\Omega_{C(b)}, D)$, so Theorem 1.13 and equation (1.12) give the following.

Corollary 1.16. *Suppose (1.11) holds for a positive integer e satisfying (1.15). Then $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ is a polynomial of degree M_f .*

The main idea in the proof of Theorem 1.13 is to relate the spectral sequence (1.6) to the spectral sequence associated to the filtration by p -divisibility on the complex $(\Omega_{C(b)}, D)$. This is accomplished by Theorem 3.1, which is applied in section 4 to compute the cohomology of $(\Omega_{C(b)}, D)$.

We conjecture that the hypothesis of Corollary 1.16 implies that all reciprocal roots of $L(\mathbf{A}^n, f; t)^{(-1)^{n+1}}$ have absolute value $q^{n/2}$. This would imply, in particular, the estimate

$$|S(\mathbf{A}^n(\mathbf{F}_{q^i}), f)| \leq M_f q^{ni/2}.$$

Typically, such results are proved by computing the corresponding l -adic cohomology groups. However, we have been unable to apply our previous method [1, 2] for calculating l -adic cohomology from p -adic cohomology because we have been unable to compute the p -divisibility of the determinant of Frobenius under the hypothesis of Corollary 1.16.

It is an interesting problem to find geometric conditions that imply (1.11) for some $e > 1$, and we plan to return to this question in a future article. As an example, we prove in section 5 the following. Make (1.3) more precise by writing

$$(1.17) \quad f = f^{(\delta)} + f^{(\delta')} + f^{(\delta'-1)} + \dots + f^{(0)},$$

where $f^{(j)}$ is homogeneous of degree j and $1 \leq \delta' \leq \delta - 1$, i. e., $f^{(\delta')}$ is the homogeneous part of second-highest degree of f .

Theorem 1.18. *Suppose that $f^{(\delta)} = f_1^{a_1} \dots f_r^{a_r}$, where for every subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, r\}$ the system of equations*

$$f_{i_1} = \dots = f_{i_k} = 0$$

defines a smooth complete intersection of codimension k in \mathbf{P}^{n-1} and, if either $k \geq 2$ or $k = 1$ and $a_{i_1} > 1$, the system of equations

$$f^{(\delta')} = f_{i_1} = \dots = f_{i_k} = 0$$

defines a smooth complete intersection of codimension $k+1$ in \mathbf{P}^{n-1} . Suppose also that $(p, \delta\delta'a_1 \dots a_r) = 1$. Then (1.11) holds for $e = \delta - \delta' + 1$.

We state here (the proof will appear elsewhere) the first such example we found. We refer to García [7] for the definitions of “weighted homogeneous” isolated singularity and “total degree” of a weighted homogeneous isolated singularity.

Theorem 1.19. *Suppose that the hypersurface $f^{(\delta)} = 0$ in \mathbf{P}^{n-1} has at worst weighted homogeneous isolated singularities, of total degrees $\delta_1, \dots, \delta_s$, and that none of these singularities lies on the hypersurface $f^{(\delta')} = 0$ in \mathbf{P}^{n-1} . Suppose also that $(p, \delta\delta'\delta_1 \dots \delta_s) = 1$. Then (1.11) holds for $e = \delta - \delta' + 1$.*

The hypothesis of Theorem 1.19 (for $\delta' = \delta - 1$) first appears in García [7]. García shows that it implies that the l -adic cohomology groups associated to the exponential sum (1.1) vanish except in degree n , where the cohomology group is pure of weight n and has dimension M_f . Thus the conclusion of Theorem 1.10 holds in this case. García’s results and Theorem 1.19 are what originally led us to suspect a result like Theorem 1.13 should hold.

2. p -ADIC COHOMOLOGY

In this section we review the basic properties of Dwork’s p -adic cohomology theory that we shall need. For a more detailed exposition of this material, we refer the reader to [2].

Let \mathbf{Q}_p be the field of p -adic numbers, ζ_p a primitive p -th root of unity, and $\Omega_1 = \mathbf{Q}_p(\zeta_p)$. The field Ω_1 is a totally ramified extension of \mathbf{Q}_p of degree $p - 1$. Let K be the unramified extension of \mathbf{Q}_p of degree a . Set $\Omega_0 = K(\zeta_p)$. The Frobenius automorphism $x \mapsto x^p$ of $\text{Gal}(\mathbf{F}_q/\mathbf{F}_p)$ lifts to a generator τ of $\text{Gal}(\Omega_0/\Omega_1) (\simeq \text{Gal}(K/\mathbf{Q}_p))$ by requiring $\tau(\zeta_p) = \zeta_p$. Let Ω be the completion of an algebraic closure of Ω_0 . Denote by “ord” the additive valuation on Ω normalized by $\text{ord } p = 1$ and by “ord $_q$ ” the additive valuation normalized by $\text{ord}_q q = 1$.

Let $E(t)$ be the Artin-Hasse exponential series:

$$E(t) = \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\right).$$

Let $\gamma \in \Omega_1$ be a solution of $\sum_{i=0}^{\infty} t^{p^i}/p^i = 0$ satisfying $\text{ord } \gamma = 1/(p-1)$ and put

$$(2.1) \quad \theta(t) = E(\gamma t) = \sum_{i=0}^{\infty} \lambda_i t^i \in \Omega_1[[t]].$$

The series $\theta(t)$ is a splitting function[6] whose coefficients satisfy

$$(2.2) \quad \text{ord } \lambda_i \geq i/(p-1).$$

We consider the following spaces of p -adic functions. Let b be a positive rational number and choose a positive integer M such that Mb/p and $M\delta/(p(p-1))$ are integers. Let π be such that

$$(2.3) \quad \pi^{M\delta} = p$$

and put $\tilde{\Omega}_1 = \Omega_1(\pi)$, $\tilde{\Omega}_0 = \Omega_0(\pi)$. The element π is a uniformizing parameter for the rings of integers of $\tilde{\Omega}_1$ and $\tilde{\Omega}_0$. We extend $\tau \in \text{Gal}(\Omega_0/\Omega_1)$ to a generator of $\text{Gal}(\tilde{\Omega}_0/\tilde{\Omega}_1)$ by requiring $\tau(\pi) = \pi$. For $u = (u_1, \dots, u_n) \in \mathbf{R}^n$, we put $|u| = u_1 + \dots + u_n$. Define

$$(2.4) \quad C(b) = \left\{ \sum_{u \in \mathbf{N}^n} A_u \pi^{Mb|u|} x^u \mid A_u \in \tilde{\Omega}_0 \text{ and } A_u \rightarrow 0 \text{ as } |u| \rightarrow \infty \right\}.$$

For $\xi = \sum_{u \in \mathbf{N}^n} A_u \pi^{Mb|u|} x^u \in C(b)$, define

$$\text{ord } \xi = \min_{u \in \mathbf{N}^n} \{\text{ord } A_u\}.$$

Given $c \in \mathbf{R}$, we put

$$C(b, c) = \{\xi \in C(b) \mid \text{ord } \xi \geq c\}.$$

Let $\hat{f} = \sum_u \hat{a}_u x^u \in K[x_1, \dots, x_n]$ be the Teichmüller lifting of the polynomial $f \in \mathbf{F}_q[x_1, \dots, x_n]$, i. e., $(\hat{a}_u)^q = \hat{a}_u$ and the reduction of \hat{f} modulo p is f . Set

$$(2.5) \quad F(x) = \prod_u \theta(\hat{a}_u x^u),$$

$$(2.6) \quad F_0(x) = \prod_{i=0}^{a-1} \prod_u \theta((\hat{a}_u x^u)^{p^i}).$$

The estimate (2.2) implies that $F \in C(b, 0)$ for all $b < 1/(p-1)$ and $F_0 \in C(b, 0)$ for all $b < p/q(p-1)$. Define an operator ψ on formal power series by

$$(2.7) \quad \psi\left(\sum_{u \in \mathbf{N}^n} A_u x^u\right) = \sum_{u \in \mathbf{N}^n} A_{pu} x^u.$$

It is clear that $\psi(C(b, c)) \subseteq C(pb, c)$. For $0 < b < p/(p-1)$, let $\alpha = \psi^a \circ F_0$ be the composition

$$C(b) \hookrightarrow C(b/q) \xrightarrow{F_0} C(b/q) \xrightarrow{\psi^a} C(b).$$

Then α is a completely continuous $\tilde{\Omega}_0$ -linear endomorphism of $C(b)$. We shall also need to consider $\beta = \tau^{-1} \circ \psi \circ F$, which is a completely continuous $\tilde{\Omega}_1$ -linear (or $\tilde{\Omega}_0$ -semilinear) endomorphism of $C(b)$. Note that $\alpha = \beta^a$.

Set $\hat{f}_i = \partial \hat{f} / \partial x_i$ and let $\gamma_l = \sum_{i=0}^l \gamma^{p^i} / p^i$. By the definition of γ , we have

$$(2.8) \quad \text{ord } \gamma_l \geq \frac{p^{l+1}}{p-1} - l - 1.$$

For $i = 1, \dots, n$, define differential operators D_i by

$$(2.9) \quad D_i = \pi^\epsilon \frac{\partial}{\partial x_i} + \pi^\epsilon H_i,$$

where $\epsilon = Mb(\delta - 1) - M\delta/(p-1)$ and

$$(2.10) \quad H_i = \sum_{l=0}^{\infty} \gamma_l p^l x_i^{p^l - 1} \hat{f}_i^{p^l}(x^{p^l}) \in C\left(b, \frac{1}{p-1} - b \frac{\delta-1}{\delta}\right)$$

for $b < p/(p-1)$. Thus D_i and “multiplication by H_i ” operate on $C(b)$ for $b < p/(p-1)$. As explained in [2], we have

$$(2.11) \quad \alpha \circ x_i D_i = q x_i D_i \circ \alpha,$$

$$(2.12) \quad \beta \circ x_i D_i = p x_i D_i \circ \beta.$$

The significance of the normalizing factor π^ϵ will be explained below.

Consider the de Rham-type complex $(\Omega_{C(b)}^\bullet, D)$, where

$$\Omega_{C(b)}^k = \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} C(b) dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

and $D : \Omega_{C(b)}^k \rightarrow \Omega_{C(b)}^{k+1}$ is defined by

$$D(\xi dx_{i_1} \wedge \dots \wedge dx_{i_k}) = \left(\sum_{i=1}^n D_i(\xi) dx_i \right) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

We extend the mapping α to a mapping $\alpha_\bullet : \Omega_{C(b)}^\bullet \rightarrow \Omega_{C(b)}^\bullet$ defined by linearity and the formula

$$\alpha_k(\xi dx_{i_1} \wedge \dots \wedge dx_{i_k}) = q^{n-k} \frac{1}{x_{i_1} \dots x_{i_k}} \alpha(x_{i_1} \dots x_{i_k} \xi) dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

Equation (2.11) implies that α_\bullet is a map of complexes. The Dwork trace formula, as formulated by Robba[11], then gives

$$(2.13) \quad L(\mathbf{A}^n, f; t) = \prod_{k=0}^n \det(I - t\alpha_k | \Omega_{C(b)}^k)^{(-1)^{k+1}}.$$

This implies

$$(2.14) \quad L(\mathbf{A}^n, f; t) = \prod_{k=0}^n \det(I - t\alpha_k | H^k(\Omega_{C(b)}^\bullet, D))^{(-1)^{k+1}},$$

where we denote the induced map on cohomology by α_k also.

The p -adic Banach space $C(b)$ has a decreasing filtration $\{\hat{F}^r C(b)\}_{r=-\infty}^\infty$ defined by setting

$$\hat{F}^r C(b) = \left\{ \sum_{u \in \mathbf{N}^n} A_u \pi^{Mb|u|} x^u \in C(b) \mid A_u \in \pi^r \mathcal{O}_{\tilde{\Omega}_0} \text{ for all } u \right\},$$

where $\mathcal{O}_{\tilde{\Omega}_0}$ denotes the ring of integers of $\tilde{\Omega}_0$. We extend this to a filtration on $\Omega_{C(b)}$ by defining

$$\hat{F}^r \Omega_{C(b)}^k = \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} \hat{F}^r C(b) dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

This filtration is exhaustive and separated, i. e.,

$$\bigcup_{r \in \mathbf{Z}} \hat{F}^r \Omega_{C(b)} = \Omega_{C(b)} \quad \text{and} \quad \bigcap_{r \in \mathbf{Z}} \hat{F}^r \Omega_{C(b)} = (0).$$

Our choice of the normalizing factor π^ϵ in (2.9) guarantees that the D_i respect this filtration, i. e., $D_i(\hat{F}^r C(b)) \subseteq \hat{F}^r C(b)$, hence $D(\hat{F}^r \Omega_{C(b)}^k) \subseteq \hat{F}^r \Omega_{C(b)}^{k+1}$. Associated to the filtered complex $(\Omega_{C(b)}, D)$ is the spectral sequence

$$(2.15) \quad \hat{E}_1^{r,s} = H^{r+s}(\hat{F}^r \Omega_{C(b)} / \hat{F}^{r+1} \Omega_{C(b)}) \Rightarrow H^{r+s}(\Omega_{C(b)}, D).$$

The notation $\hat{E}_t^{r,s}$ does not express the dependence of this spectral sequence on the choice of b , however, this should not cause confusion. We shall prove Theorem 1.13 by analyzing this spectral sequence.

For notational convenience we define an “exterior derivative” $d : \Omega_{C(b)}^k \rightarrow \Omega_{C(b)}^{k+1}$. It is characterised by $\tilde{\Omega}_0$ -linearity and the formula

$$d(\xi dx_{i_1} \wedge \dots \wedge dx_{i_k}) = \left(\sum_{i=1}^n \frac{\partial \xi}{\partial x_i} dx_i \right) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

Although we use the same symbol “ d ” for the exterior derivative on both $\Omega_{C(b)}$ and $\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}$, its meaning will be clear from the context.

It will also be convenient to define a “reduction map” $\text{red} : \hat{F}^0 \Omega_{C(b)}^m \rightarrow \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$ as follows. For

$$\omega = \sum_{1 \leq i_1 < \dots < i_m \leq n} \left(\sum_{u \in \mathbf{N}^n} A_u(i_1, \dots, i_m) \pi^{Mb|u|} x^u \right) dx_{i_1} \wedge \dots \wedge dx_{i_m} \in \hat{F}^0 \Omega_{C(b)}^m,$$

define

$$(2.16) \quad \text{red}(\omega) = \sum_{1 \leq i_1 < \dots < i_m \leq n} \left(\sum_{u \in \mathbf{N}^n} \bar{A}_u(i_1, \dots, i_m) x^u \right) dx_{i_1} \wedge \dots \wedge dx_{i_m} \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m,$$

where $\bar{A}_u(i_1, \dots, i_m) \in \mathbf{F}_q$ is the reduction mod π of $A_u(i_1, \dots, i_m) \in \mathcal{O}_{\tilde{\Omega}_0}$. Note that the inner sum in (2.16) is finite by the definition of $C(b)$ (equation (2.4)).

3. RELATION BETWEEN THE SPECTRAL SEQUENCES $E_t^{r,s}$ AND $\hat{E}_t^{r,s}$

The key to our computation of p -adic cohomology is the following.

Theorem 3.1. *Fix a positive integer e , let $m \in \{0, 1, \dots, n\}$, and let b be a rational number satisfying (1.14). Let $\{\phi_i\}_{i \in I} \subseteq \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$, I an index set, be a set of homogeneous m -forms such that for each $r \geq 0$, the classes $\{[\phi_i]\}_{\deg \phi_i = r}$ form a basis for $E_e^{r, m-r}$ as \mathbf{F}_q -vector space. Let $\{\tilde{\phi}_i\}_{i \in I} \subseteq \hat{F}^0 \Omega_{C(b)}^m$ be a set of m -forms satisfying $\text{red}(\tilde{\phi}_i) = \phi_i$ for all $i \in I$. Then for each $r \in \mathbf{Z}$, the classes $\{[\pi^r \tilde{\phi}_i]\}_{i \in I}$ form a basis for $\hat{E}_{Mb(e-1)+1}^{r, m-r}$ as \mathbf{F}_q -vector space.*

Remark. We observe for later reference that (1.14) implies $\text{ord}_p \pi^{Mb(e-1)} < 1$.

Proof. Consider the operator $D_i = \pi^\epsilon \partial / \partial x_i + \pi^\epsilon H_i$. The terms of degree $< \delta - e$ in $\pi^\epsilon H_i$ lie in $\hat{F}^{Mbe} C(b)$. The upper bound on b given by (1.14) guarantees that the terms of degree $> \delta - 1$ in $\pi^\epsilon H_i$ lie in $\hat{F}^{Mb(e-1)+1} C(b)$. The lower bound on b given by (1.14) guarantees that $\pi^\epsilon \partial / \partial x_i$ maps $\hat{F}^0 C(b)$ into $\hat{F}^{Mb(e-1)+1} C(b)$. We thus have for $\xi \in \hat{F}^0 C(b)$,

$$(3.2) \quad D_i(\xi) - \pi^{Mb(\delta-1)} \left(\sum_{j=0}^{e-1} \frac{\partial \hat{f}^{(\delta-j)}}{\partial x_i} \right) \xi \in \hat{F}^{Mb(e-1)+1} C(b).$$

It follows that if $\omega \in \hat{F}^r \Omega_{C(b)}^k$, $0 \leq k \leq n$, then

$$(3.3) \quad D(\omega) - \pi^{Mb(\delta-1)} \left(\sum_{j=0}^{e-1} d\hat{f}^{(\delta-j)} \right) \wedge \omega \in \hat{F}^{r+Mb(e-1)+1} \Omega_{C(b)}^k.$$

Note that

$$(3.4) \quad \pi^{Mb(\delta-j-1)} d\hat{f}^{(\delta-j)} \in \hat{F}^0 \Omega_{C(b)}^1 \quad \text{for } j = 0, 1, \dots, e-1.$$

We recall the definition of $\hat{E}_t^{r,s}$. Put

$$Z_t^{r,s} = \{\omega \in \hat{F}^r \Omega_{C(b)}^{r+s} \mid D(\omega) \in \hat{F}^{r+t} \Omega_{C(b)}^{r+s+1}\}.$$

Then

$$\hat{E}_t^{r,s} = \frac{Z_t^{r,s} + \hat{F}^{r+1} \Omega_{C(b)}^{r+s}}{D(Z_{t-1}^{r-t+1, s+t-2}) + \hat{F}^{r+1} \Omega_{C(b)}^{r+s}}.$$

Theorem 3.1 asserts that if $\omega \in Z_{Mb(e-1)+1}^{r, m-r}$, then there exist a finite subset $I_0 \subseteq I$, a collection $\{c_i\}_{i \in I_0} \subseteq \mathcal{O}_{\hat{\Omega}_0}$ uniquely determined mod π , and

$$\xi \in Z_{Mb(e-1)}^{r-Mb(e-1), m-r-1+Mb(e-1)}$$

such that

$$\omega \equiv \sum_{i \in I_0} c_i \pi^r \tilde{\phi}_i + D(\xi) \pmod{\hat{F}^{r+1} \Omega_{C(b)}^m}.$$

Since $\pi^r \hat{F}^0 = \hat{F}^r$, we may reduce to the case $r = 0$ by multiplication by π^{-r} . So let $\omega \in \hat{F}^0 \Omega_{C(b)}^m$ be such that

$$(3.5) \quad D(\omega) \in \hat{F}^{Mb(e-1)+1} \Omega_{C(b)}^{m+1}.$$

We must show that there exist $\{c_i\}_{i \in I_0} \subseteq \mathcal{O}_{\hat{\Omega}_0}$ uniquely determined mod π and $\xi \in \hat{F}^{-Mb(e-1)} \Omega_{C(b)}^{m-1}$ such that

$$(3.6) \quad \omega \equiv \sum_{i \in I_0} c_i \tilde{\phi}_i + D(\xi) \pmod{\hat{F}^1 \Omega_{C(b)}^m}.$$

In view of (3.3), (3.5) is equivalent to the condition

$$(3.7) \quad \left(\sum_{j=0}^{e-1} \pi^{Mb(\delta-1)} d\hat{f}^{(\delta-j)} \right) \wedge \omega \in \hat{F}^{Mb(e-1)+1} \Omega_{C(b)}^{m+1}$$

and (3.6) is equivalent to the condition

$$(3.8) \quad \omega \equiv \sum_{i \in I_0} c_i \tilde{\phi}_i + \left(\sum_{j=0}^{e-1} \pi^{Mb(\delta-1)} d\hat{f}^{(\delta-j)} \right) \wedge \xi \pmod{\hat{F}^1 \Omega_{C(b)}^m}.$$

This reduces the proof of Theorem 3.1 to showing that, given $\omega \in \hat{F}^0 \Omega_{C(b)}^m$ satisfying (3.7), there exist a finite set $\{c_i\}_{i \in I_0} \subseteq \mathcal{O}_{\tilde{\Omega}_0}$, uniquely determined mod π , and $\xi \in \hat{F}^{-Mb(e-1)} \Omega_{C(b)}^{m-1}$ satisfying (3.8).

Equations (3.4) and (3.7) imply that for $i = 0, 1, \dots, e-2$ we have

$$\left(\sum_{j=0}^i \pi^{Mb(\delta-1)} d\hat{f}^{(\delta-j)} \right) \wedge \omega \in \hat{F}^{Mb(i+1)} \Omega_{C(b)}^{m+1}.$$

We may thus define $\eta_i \in \hat{F}^0 \Omega_{C(b)}^{m+1}$ for $i = -1, 0, 1, \dots, e-2$ recursively as follows. Set $\eta_{-1} = 0$ and for $i = 0, 1, \dots, e-2$ define η_i by the formula

$$(3.9) \quad \eta_{i-1} + \pi^{Mb(\delta-1-i)} d\hat{f}^{(\delta-i)} \wedge \omega = \pi^{Mb} \eta_i.$$

From (3.7) we also get that

$$(3.10) \quad \eta_{e-2} + \pi^{Mb(\delta-e)} d\hat{f}^{(\delta-e+1)} \wedge \omega \equiv 0 \pmod{\hat{F}^1 \Omega_{C(b)}^{m+1}}.$$

Write

$$(3.11) \quad \omega = \sum_{k=0}^{\infty} \pi^{Mb k} \omega^{(k)}$$

$$(3.12) \quad \eta_i = \sum_{k=0}^{\infty} \pi^{Mb k} \eta_i^{(k)},$$

where $\omega^{(k)}$ (resp. $\eta_i^{(k)}$) is an m -form (resp. $(m+1)$ -form) whose coefficients are homogeneous polynomials of degree k with coefficients in $\mathcal{O}_{\tilde{\Omega}_0}$. Substituting (3.11) and (3.12) into (3.9) and cancelling a power of π gives

$$(3.13) \quad \eta_{i-1}^{(k+\delta-1-i)} + d\hat{f}^{(\delta-i)} \wedge \omega^{(k)} = \pi^{Mb} \eta_i^{(k+\delta-1-i)}$$

for $i = 0, 1, \dots, e-2$ and all k . From (3.10) we also have

$$(3.14) \quad \eta_{e-2}^{(k+\delta-1-i)} + d\hat{f}^{(\delta-e+1)} \wedge \omega^{(k)} \equiv 0 \pmod{\pi}.$$

Since $\tilde{\Omega}_0 = K(\pi)$, where K is an unramified extension of \mathbf{Q}_p and π is an $(M\delta)$ -th root of p , we may write

$$(3.15) \quad \omega^{(k)} = \sum_{l=0}^{\infty} \pi^l \omega_l^{(k)},$$

where $\omega_l^{(k)}$ is an m -form whose coefficients are homogeneous polynomials with coefficients in \mathcal{O}_K , the ring of integers of K . For $i = 0, 1, \dots, e-2$ and $k \geq 0$, put

$$(3.16) \quad \zeta_i^{(k)} = \sum_{l=0}^{\infty} \pi^l \omega_{l+Mb(i+1)}^{(k)}.$$

Lemma 3.17. *For $i = 0, 1, \dots, e-2$ and $k \geq 0$, we have*

$$(3.18) \quad \eta_i^{(k+\delta-1-i)} \equiv \sum_{j=0}^i d\hat{f}^{(\delta-j)} \wedge \zeta_{i-j}^{(k-i+j)} \pmod{\pi^{Mb(e-i-2)+1}}.$$

Remark. The congruence (3.18) is to be interpreted as meaning that the difference between the two sides is an $(m+1)$ -form all of whose coefficients are homogeneous polynomials of degree $k+\delta-1-i$ with coefficients in $\pi^{Mb(e-i-2)+1}\mathcal{O}_{\hat{\Omega}_0}$.

Proof. The assertion is vacuous for $e = 1$, so we assume $e \geq 2$. We fix $k+\delta-1-i$ and prove (3.18) by induction on i . By (3.13) with $i = 0$ and (3.15) we have

$$(3.19) \quad d\hat{f}^{(\delta)} \wedge \sum_{l=0}^{\infty} \pi^l \omega_l^{(k)} = \pi^{Mb} \eta_0^{(k+\delta-1)},$$

which implies

$$d\hat{f}^{(\delta)} \wedge \omega_0^{(k)} \equiv 0 \pmod{\pi}.$$

But since the left-hand side has coefficients in \mathcal{O}_K , this congruence must actually hold mod p . Suppose we have proved

$$(3.20) \quad d\hat{f}^{(\delta)} \wedge \omega_l^{(k)} \equiv 0 \pmod{p}$$

for $l = 0, 1, \dots, L$, where $0 \leq L < Mb-1$. Since $\pi^{Mb(e-1)+1}$ divides p (see the remark following Theorem 3.1) it follows from (3.19) that

$$d\hat{f}^{(\delta)} \wedge \sum_{l=L+1}^{\infty} \pi^l \omega_l^{(k)} \equiv 0 \pmod{\pi^{Mb}}.$$

Dividing by π^{L+1} gives

$$d\hat{f}^{(\delta)} \wedge \omega_{L+1}^{(k)} \equiv 0 \pmod{\pi},$$

and again the left-hand side has coefficients in \mathcal{O}_K , so this congruence holds mod p . This proves that (3.20) holds for $0 \leq l \leq Mb-1$. Equation (3.19) then implies

$$d\hat{f}^{(\delta)} \wedge \sum_{l=Mb}^{\infty} \pi^l \omega_l^{(k)} \equiv \pi^{Mb} \eta_0^{(k+\delta-1)} \pmod{\pi^{Mb(e-1)+1}}.$$

Dividing this congruence by π^{Mb} gives (3.18) for $i = 0$.

Suppose (3.18) holds for some i , $0 \leq i < e-2$. We prove (3.18) for $i+1$. From (3.13) and the induction hypothesis we have

$$\begin{aligned} \sum_{j=0}^i d\hat{f}^{(\delta-j)} \wedge \zeta_{i-j}^{(k+j-i)} + d\hat{f}^{(\delta-i-1)} \wedge \omega^{(k+1)} &\equiv \\ &\pi^{Mb} \eta_{i+1}^{(k+\delta-1-i)} \pmod{\pi^{Mb(e-i-2)+1}}. \end{aligned}$$

Using (3.15) and (3.16), this is equivalent to

$$(3.21) \quad \sum_{j=0}^i d\hat{f}^{(\delta-j)} \wedge \left(\sum_{l=0}^{\infty} \pi^l \omega_{l+Mb(i-j+1)}^{(k+j-i)} \right) + d\hat{f}^{(\delta-i-1)} \wedge \left(\sum_{l=0}^{\infty} \pi^l \omega_l^{(k+1)} \right) \equiv \pi^{Mb} \eta_{i+1}^{(k+\delta-1-i)} \pmod{\pi^{Mb(e-i-2)+1}}.$$

Since $e - i > 2$, this implies

$$\sum_{j=0}^i d\hat{f}^{(\delta-j)} \wedge \omega_{Mb(i-j+1)}^{(k+j-i)} + d\hat{f}^{(\delta-i-1)} \wedge \omega_0^{(k+1)} \equiv 0 \pmod{\pi},$$

and since the left-hand side has coefficients in \mathcal{O}_K , this congruence holds mod p . Arguing by induction on l exactly as in the case $i = 0$ gives

$$\sum_{j=0}^i d\hat{f}^{(\delta-j)} \wedge \omega_{l+Mb(i-j+1)}^{(k+j-i)} + d\hat{f}^{(\delta-i-1)} \wedge \omega_l^{(k+1)} \equiv 0 \pmod{p}$$

for $0 \leq l \leq Mb - 1$. Equation (3.21) then implies

$$\begin{aligned} \sum_{j=0}^i d\hat{f}^{(\delta-j)} \wedge \left(\sum_{l=Mb}^{\infty} \pi^l \omega_{l+Mb(i-j+1)}^{(k+j-i)} \right) + d\hat{f}^{(\delta-i-1)} \wedge \left(\sum_{l=Mb}^{\infty} \pi^l \omega_l^{(k+1)} \right) \equiv \\ \pi^{Mb} \eta_{i+1}^{(k+\delta-1-i)} \pmod{\pi^{Mb(e-i-2)+1}}. \end{aligned}$$

Dividing by π^{Mb} now gives

$$\begin{aligned} \eta_{i+1}^{(k+\delta-1-i)} \equiv \sum_{j=0}^i d\hat{f}^{(\delta-j)} \wedge \left(\sum_{l=0}^{\infty} \pi^l \omega_{l+Mb(i-j+2)}^{(k+j-i)} \right) + d\hat{f}^{(\delta-i-1)} \wedge \left(\sum_{l=0}^{\infty} \pi^l \omega_{l+Mb}^{(k+1)} \right) \\ \pmod{\pi^{Mb(e-i-3)+1}}. \end{aligned}$$

Taking into account (3.16), we see that this is just (3.18) for $i + 1$.

Combining Lemma 3.17 with equations (3.13) and (3.14) gives

$$(3.22) \quad \sum_{j=0}^{i-1} d\hat{f}^{(\delta-j)} \wedge \zeta_{i-1-j}^{(k+j-i)} + d\hat{f}^{(\delta-i)} \wedge \omega^{(k)} \equiv 0 \pmod{\pi}$$

for $i = 0, 1, \dots, e - 1$.

For $i \in I$, write

$$\tilde{\phi}_i = \sum_{k=0}^{\infty} \pi^{Mbk} \tilde{\phi}_i^{(k)},$$

where $\tilde{\phi}_i^{(k)}$ is an m -form whose coefficients are homogeneous polynomials of degree k with coefficients in $\mathcal{O}_{\tilde{\Omega}_0}$. Our hypothesis that $\text{red}(\tilde{\phi}_i) = \phi_i$ implies that if $\deg_{\text{coeff}} \phi_i = k_0$, then ϕ_i is the reduction mod π of $\tilde{\phi}_i^{(k_0)}$, while for $k' \neq k_0$, the reduction mod π of $\tilde{\phi}_i^{(k')}$ is 0.

Lemma 3.23. *For each k there exist $\{c_i\}_{\deg_{\text{coeff}} \phi_i = k} \subseteq \mathcal{O}_K$ and $\{\xi_j^{(k-\delta+j)}\}_{j=1}^e \subseteq \Omega_{C(b)}^{m-1}$, where $\xi_j^{(k-\delta+j)}$ is an $(m-1)$ -form whose coefficients are homogeneous polynomials of degree $k - \delta + j$ with coefficients in \mathcal{O}_K , such that*

$$(3.24) \quad \omega^{(k)} \equiv \sum_{\deg_{\text{coeff}} \phi_i = k} c_i \tilde{\phi}_i^{(k)} + \sum_{j=0}^{e-1} d\hat{f}^{(\delta-j)} \wedge \xi_{j+1}^{(k-\delta+j+1)} \pmod{\pi}$$

and such that

$$(3.25) \quad \sum_{j=0}^i d\hat{f}^{(\delta-j)} \wedge \xi_{j+e-i}^{(k-\delta+j+e-i)} \equiv 0 \pmod{p}$$

for $i = 0, 1, \dots, e-2$.

Remark. By the definition of $C(b)$, there exists a positive integer k_ω such that $\omega^{(k)} \equiv 0 \pmod{\pi}$ for all $k > k_\omega$. For $k > k_\omega$ we take all c_i and all $\xi_j^{(k-\delta+j)}$ equal to 0, which satisfies the conclusion of the lemma.

Proof. Let $\bar{\omega}^{(k)} \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$ (resp. $\bar{\zeta}_j^{(k)} \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$) be the reduction mod π of $\omega^{(k)}$ (resp. $\zeta_j^{(k)}$). Then (3.22) implies

$$\sum_{j=0}^{i-1} df^{(\delta-j)} \wedge \bar{\zeta}_{i-1-j}^{(k-i+j)} + df^{(\delta-i)} \wedge \bar{\omega}^{(k)} = 0$$

for $i = 0, 1, \dots, e-1$. The definition of $E_e^{r,m-r}$ and the hypothesis that $\{\phi_i\}_{\deg \phi_i=r}$ spans $E_e^{r,m-r}$ imply that there exist $\{\bar{c}_i\}_{\deg_{\text{coeff}} \phi_i=k} \subseteq \mathbf{F}_q$ and $\{\bar{\xi}_j^{(k-\delta+j)}\}_{j=1}^e \subseteq \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{m-1}$ such that

$$(3.26) \quad \bar{\omega}^{(k)} = \sum_{\deg_{\text{coeff}} \phi_i=k} \bar{c}_i \phi_i + \sum_{j=0}^{e-1} df^{(\delta-j)} \wedge \bar{\xi}_{j+1}^{(k-\delta+j+1)}$$

and such that

$$(3.27) \quad \sum_{j=0}^i df^{(\delta-j)} \wedge \bar{\xi}_{j+e-i}^{(k+j+e-i-\delta)} = 0$$

for $i = 0, 1, \dots, e-2$.

Let $c_i \in \mathcal{O}_K$ be any lifting of $\bar{c}_i \in \mathbf{F}_q$ and let $\xi_j^{(k+j-\delta)} \in \Omega_{C(b)}^{m-1}$ be any lifting of $\bar{\xi}_j^{(k+j-\delta)}$ that has coefficients in \mathcal{O}_K . Thus $\xi_j^{(k+j-\delta)}$ is an $(m-1)$ -form whose coefficients are homogeneous polynomials of degree $k+j-\delta$ with coefficients in \mathcal{O}_K . Then (3.26) and (3.27) imply (3.24) and (3.25), the congruence (3.25) holding mod p rather than just mod π because the left-hand side has coefficients in \mathcal{O}_K . This completes the proof of Lemma 3.23.

We continue with the proof of Theorem 3.1. Put

$$\xi_i = \sum_{k=0}^{\infty} \pi^{Mb(k-\delta+i)} \xi_i^{(k-\delta+i)} \in \hat{F}^0 \Omega_{C(b)}^{m-1}$$

for $i = 1, 2, \dots, e$. Let $I_0 \subseteq I$ be the (finite) subset consisting of all indices i such that $\deg_{\text{coeff}} \phi_i \leq k_\omega$. Then (3.11), (3.24), and the remark following Lemma 3.23 imply

$$(3.28) \quad \omega \equiv \sum_{i \in I_0} c_i \tilde{\phi}_i + \sum_{j=0}^{e-1} \pi^{Mb(\delta-j-1)} df^{(\delta-j)} \wedge \xi_{j+1} \pmod{\hat{F}^1 \Omega_{C(b)}^m}.$$

Since p is divisible by $\pi^{Mb(e-1)+1}$, (3.25) implies

$$(3.29) \quad \sum_{j=0}^i \pi^{Mb(\delta-j-1)} df^{(\delta-j)} \wedge \xi_{j+e-i} \equiv 0 \pmod{\hat{F}^{Mb(e-1)+1} \Omega_{C(b)}^m}$$

for $i = 0, 1, \dots, e-2$. Put

$$(3.30) \quad \xi = \sum_{j=0}^{e-1} \pi^{-Mbj} \xi_{j+1} \in \hat{F}^{-Mb(e-1)} \Omega_{C(b)}^{m-1}.$$

With this choice of ξ , the right-hand side of (3.8) becomes

$$(3.31) \quad \sum_{i \in I_0} c_i \tilde{\phi}_i + \left(\sum_{i=0}^{e-1} \pi^{Mbi} (\pi^{Mb(\delta-i-1)} d\hat{f}^{(\delta-i)}) \right) \wedge \left(\sum_{j=0}^{e-1} \pi^{-Mbj} \xi_{j+1} \right).$$

When (3.31) is expanded, the wedge product of a pair of terms with $i > j$ lies in $\hat{F}^{Mb} \Omega_{C(b)}^m$. Putting $k = j - i$ when $j \geq i$, we see that (3.31) is congruent mod $\hat{F}^{Mb} \Omega_{C(b)}^m$ to

$$(3.32) \quad \sum_{i \in I_0} c_i \tilde{\phi}_i + \sum_{k=0}^{e-1} \pi^{-Mbk} \sum_{l=0}^{e-1-k} \pi^{Mb(\delta-l-1)} d\hat{f}^{(\delta-l)} \wedge \xi_{l+k+1}.$$

For $k = 1, \dots, e-1$, the corresponding summand of (3.32) lies in $\hat{F}^1 \Omega_{C(b)}^m$ by (3.29), hence (3.32) is congruent mod $\hat{F}^1 \Omega_{C(b)}^m$ to

$$\sum_{i \in I_0} c_i \tilde{\phi}_i + \sum_{l=0}^{e-1} \pi^{Mb(\delta-l-1)} d\hat{f}^{(\delta-l)} \wedge \xi_{l+1}.$$

But this is $\equiv \omega \pmod{\hat{F}^1 \Omega_{C(b)}^m}$ by (3.28). Thus congruence (3.8) holds.

To complete the proof of Theorem 3.1, it remains to show that the c_i are uniquely determined mod π . Suppose there exist a finite subset $I_0 \subseteq I$, $\{c_i\}_{i \in I_0} \subseteq \mathcal{O}_{\hat{\Omega}_0}$ and $\xi \in \hat{F}^{-Mb(e-1)} \Omega_{C(b)}^{m-1}$ such that

$$(3.33) \quad \sum_{i \in I_0} c_i \tilde{\phi}_i \equiv \sum_{j=0}^{e-1} \pi^{Mb(\delta-j)} d\hat{f}^{(\delta-j)} \wedge \xi \pmod{\hat{F}^1 \Omega_{C(b)}^m}.$$

We must show $c_i \equiv 0 \pmod{\pi}$ for all $i \in I_0$. Write

$$(3.34) \quad \xi = \pi^{-Mb(e-1)} \sum_{k=0}^{\infty} \pi^{Mbk} \xi^{(k)},$$

where the coefficients of $\xi^{(k)} \in \Omega_{C(b)}^{m-1}$ are homogeneous forms of degree k with coefficients in $\mathcal{O}_{\hat{\Omega}_0}$. Substituting (3.34) into (3.33) and cancelling π^{Mbk} gives

$$(3.35) \quad \sum_{\substack{i \in I_0 \\ \deg_{\text{coeff}} \phi_i = k}} c_i \tilde{\phi}_i^{(k)} \equiv \sum_{j=0}^{e-1} \pi^{Mb(j-e+1)} d\hat{f}^{(\delta-j)} \wedge \xi^{(k-\delta+j+1)} \pmod{\pi}$$

for all k . We may thus define $\eta_i \in \Omega_{C(b)}^{m-1}$ for $i = -1, 0, 1, \dots, e-2$ recursively as follows. Put $\eta_{-1} = 0$ and define η_i for $i = 0, 1, \dots, e-2$ by

$$(3.36) \quad \eta_{i-1}^{(k)} + d\hat{f}^{(\delta-i)} \wedge \xi^{(k-\delta+i+1)} = \pi^{Mb} \eta_i^{(k)}.$$

We also have from (3.35) that

$$(3.37) \quad \eta_{e-2}^{(k)} + d\hat{f}^{(\delta-e+1)} \wedge \xi^{(k-\delta+e)} \equiv \sum_{\substack{i \in I_0 \\ \deg_{\text{coeff}} \phi_i = k}} c_i \tilde{\phi}_i^{(k)} \pmod{\pi}.$$

By a shift of index, (3.36) and (3.37) become

$$(3.38) \quad \eta_{i-1}^{(k+\delta-1-i)} + d\hat{f}^{(\delta-i)} \wedge \xi^{(k)} = \pi^{Mb} \eta_i^{(k+\delta-1-i)}$$

for $i = 0, 1, \dots, e-2$ and

$$(3.39) \quad \eta_{e-2}^{(k+\delta-e)} + df^{(\delta-e+1)} \wedge \xi^{(k)} \equiv \sum_{\substack{i \in I_0 \\ \deg_{\text{coeff}} \phi_i = k+\delta-e}} c_i \tilde{\phi}_i^{(k+\delta-e)} \pmod{\pi}.$$

Write $\xi^{(k)} = \sum_{l=0}^{\infty} \pi^l \xi_l^{(k)}$, where $\xi_l^{(k)}$ is an $(m-1)$ -form whose coefficients are homogeneous polynomials of degree k with coefficients in \mathcal{O}_K . Put

$$\rho_i^{(k)} = \sum_{l=0}^{\infty} \pi^l \xi_{l+Mb(i+1)}^{(k)}.$$

Since (3.38) is identical in form to (3.13), we may apply Lemma 3.17 to conclude that

$$(3.40) \quad \eta_i^{(k+\delta-1-i)} \equiv \sum_{j=0}^i df^{(\delta-j)} \wedge \rho_{i-1-j}^{(k-i+j)} \pmod{\pi^{Mb(e-i-2)+1}}.$$

Substitution into (3.38) and (3.39) then gives

$$(3.41) \quad \sum_{j=0}^{i-1} df^{(\delta-j)} \wedge \rho_{i-1-j}^{(k+j-i)} + df^{(\delta-i)} \wedge \xi^{(k)} \equiv 0 \pmod{\pi}$$

for $i = 0, 1, \dots, e-2$ and

$$(3.42) \quad \sum_{j=0}^{e-2} df^{(\delta-j)} \wedge \rho_{e-2-j}^{(k+j-e-1)} + df^{(\delta-e+1)} \wedge \xi^{(k)} \equiv \sum_{\substack{i \in I_0 \\ \deg_{\text{coeff}} \phi_i = k+\delta-e}} c_i \tilde{\phi}_i^{(k+\delta-e)} \pmod{\pi}.$$

Letting $\bar{\xi}^{(k)} \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{m-1}$ (resp. $\bar{\rho}_{i-1-j}^{(k+j-i)} \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{m-1}$) be the reduction mod π of $\xi^{(k)}$ (resp. $\rho_{i-1-j}^{(k+j-i)}$), we see that equations (3.41) and (3.42) imply

$$(3.43) \quad \sum_{j=0}^{i-1} df^{(\delta-j)} \wedge \bar{\rho}_{i-1-j}^{(k+j-i)} + df^{(\delta-i)} \wedge \bar{\xi}^{(k)} = 0$$

for $i = 0, 1, \dots, e-2$ and

$$(3.44) \quad \sum_{j=0}^{e-2} df^{(\delta-j)} \wedge \bar{\rho}_{e-2-j}^{(k+j-e-1)} + df^{(\delta-e+1)} \wedge \bar{\xi}^{(k)} = \sum_{\substack{i \in I_0 \\ \deg_{\text{coeff}} \phi_i = k+\delta-e}} \bar{c}_i \phi_i,$$

where $\bar{c}_i \in \mathbf{F}_q$ denotes the reduction mod π of c_i . The definition of $E_e^{r,m-r}$ and the hypothesis that $\{\phi_i\}_{\deg \phi_i=r}$ is linearly independent in $E_e^{r,m-r}$ then imply that $\bar{c}_i = 0$ for all i . This completes the proof of Theorem 3.1.

4. COMPUTATION OF p -ADIC COHOMOLOGY

We derive a series of corollaries to Theorem 3.1

Corollary 4.1. *In addition to the hypothesis of Theorem 3.1, assume that the index set I is finite, i. e.,*

$$\dim_{\mathbf{F}_q} \left(\bigoplus_{r=0}^{\infty} E_e^{r, m-r} \right) < \infty.$$

Then the cohomology classes $\{[\tilde{\phi}_i]\}_{i \in I}$ span $H^m(\Omega_{C(b)}^m, D)$ as $\tilde{\Omega}_0$ -vector space.

Proof. Let $\omega \in \Omega_{C(b)}^m$. Without loss of generality, we may assume $\omega \in \hat{F}^0 \Omega_{C(b)}^m$. By Theorem 3.1 (see equation (3.6)), there exist

$$\{c_i^{(0)}\}_{i \in I} \subseteq \mathcal{O}_{\tilde{\Omega}_0}, \quad \omega_0 \in \hat{F}^0 \Omega_{C(b)}^m, \quad \xi_0 \in \hat{F}^{-Mb(e-1)} \Omega_{C(b)}^{m-1}$$

such that

$$(4.2) \quad \omega = \pi \omega_0 + \sum_{i \in I} c_i^{(0)} \tilde{\phi}_i + D(\xi_0).$$

Suppose that for some $t \geq 0$ we have found

$$\{c_i^{(t)}\}_{i \in I} \subseteq \mathcal{O}_{\tilde{\Omega}_0}, \quad \omega_t \in \hat{F}^0 \Omega_{C(b)}^m, \quad \xi_t \in \hat{F}^{-Mb(e-1)} \Omega_{C(b)}^{m-1}$$

such that

$$(4.3) \quad \omega = \pi^{t+1} \omega_t + \sum_{i \in I} c_i^{(t)} \tilde{\phi}_i + D(\xi_t)$$

and such that

$$c_i^{(t)} - c_i^{(t-1)} \in \pi^t \mathcal{O}_{\tilde{\Omega}_0}, \quad \xi_t - \xi_{t-1} \in \hat{F}^{-Mb(e-1)+t} \Omega_{C(b)}^{m-1}.$$

Applying Theorem 3.1 to ω_t , we see that there exist

$$\{\tilde{c}_i^{(t)}\}_{i \in I} \subseteq \mathcal{O}_{\tilde{\Omega}_0}, \quad \omega_{t+1} \in \hat{F}^0 \Omega_{C(b)}^m, \quad \xi'_t \in \hat{F}^{-Mb(e-1)} \Omega_{C(b)}^{m-1}$$

such that

$$(4.4) \quad \omega_t = \pi \omega_{t+1} + \sum_{i \in I} \tilde{c}_i^{(t)} \tilde{\phi}_i + D(\xi'_t).$$

Put $\xi_{t+1} = \xi_t + \pi^{t+1} \xi'_t$, $c_i^{(t+1)} = c_i^{(t)} + \pi^{t+1} \tilde{c}_i^{(t)}$. Substituting (4.4) into (4.3) gives

$$(4.5) \quad \omega = \pi^{t+2} \omega_{t+1} + \sum_{i \in I} c_i^{(t+1)} \tilde{\phi}_i + D(\xi_{t+1})$$

with

$$c_i^{(t+1)} - c_i^{(t)} \in \pi^{t+1} \mathcal{O}_{\tilde{\Omega}_0}, \quad \xi_{t+1} - \xi_t \in \hat{F}^{-Mb(e-1)+t+1} \Omega_{C(b)}^{m-1}.$$

It follows that $\{\xi_t\}_{t=0}^{\infty}$ converges to an element $\xi \in \hat{F}^{-Mb(e-1)} \Omega_{C(b)}^{m-1}$ and $\{c_i^{(t)}\}_{t=0}^{\infty}$ converges to an element $c_i \in \mathcal{O}_{\tilde{\Omega}_0}$ for $i \in I$ satisfying

$$\omega = \sum_{i \in I} c_i \tilde{\phi}_i + D(\xi).$$

This establishes the corollary.

Corollary 4.6. *Suppose there exist e, m such that $E_e^{r, m-r} = 0$ for all $r \geq 0$. Then for every rational number b satisfying (1.14), we have $H^m(\Omega_{C(b)}^m, D) = 0$.*

Proof. The hypothesis of Theorem 3.1 is satisfied with $I = \emptyset$, so the conclusion follows immediately from Corollary 4.1.

To ensure that the cohomology classes $\{[\tilde{\phi}_i]\}_{i \in I}$ form a basis, we need an additional hypothesis.

Corollary 4.7. *In addition to the hypothesis of Corollary 4.1, suppose that for all $r \in \mathbf{Z}$*

$$(4.8) \quad \hat{E}_{Mb(e-1)+1}^{r,m-r} = \hat{E}_{Mb(e-1)+2}^{r,m-r} = \cdots.$$

Then the set $\{[\tilde{\phi}_i]\}_{i \in I}$ is a basis for $H^m(\Omega_{C(b)}^\bullet, D)$.

Proof. By Corollary 4.1, we only need to check linear independence. Suppose we had a relation

$$(4.9) \quad \sum_{i \in I} c_i \tilde{\phi}_i = D(\xi),$$

where $\{c_i\}_{i \in I} \subseteq \tilde{\Omega}_0$ and $\xi \in \Omega_{C(b)}^{m-1}$. If the c_i were not all zero, then after multiplication by a suitable power of π we may assume that $c_i \in \mathcal{O}_{\tilde{\Omega}_0}$ for all i and $c_i \notin \pi \mathcal{O}_{\tilde{\Omega}_0}$ for some i . Thus the left-hand side of (4.9) lies in $\hat{F}^0 \Omega_{C(b)}^m$ but not in $\hat{F}^1 \Omega_{C(b)}^m$. Since the filtration \hat{F}^\bullet is exhaustive, there exists $r \geq 0$ such that

$$\xi \in \hat{F}^{-Mb(e-1)-r} \Omega_{C(b)}^{m-1}.$$

Equation (4.9) then says that $[\sum_{i \in I} c_i \tilde{\phi}_i] = 0$ in $\hat{E}_{Mb(e-1)+r+1}^{0,m}$. By (4.8), we have $[\sum_{i \in I} c_i \tilde{\phi}_i] = 0$ in $\hat{E}_{Mb(e-1)+1}^{0,m}$. Theorem 3.1 now implies $c_i \equiv 0 \pmod{\pi}$ for all i , a contradiction. Thus $c_i = 0$ for all i .

Corollary 4.10. *In addition to the hypothesis of Corollary 4.1, suppose that*

$$(4.11) \quad E_e^{r,m-1-r} = E_e^{r,m+1-r} = 0$$

for all $r \geq 0$. Then the set $\{[\tilde{\phi}_i]\}_{i \in I}$ is a basis for $H^m(\Omega_{C(b)}^\bullet, D)$. In particular,

$$\dim_{\tilde{\Omega}_0} H^m(\Omega_{C(b)}^\bullet, D) = \dim_{\mathbf{F}_q} \left(\bigoplus_{r=0}^{\infty} E_e^{r,m-r} \right).$$

Proof. Condition (4.11) and Theorem 3.1 imply that

$$\hat{E}_{Mb(e-1)+1}^{r,m-1-r} = \hat{E}_{Mb(e-1)+1}^{r,m+1-r} = 0$$

for all $r \in \mathbf{Z}$. General properties of spectral sequences then imply that (4.8) holds, and we conclude by Corollary 4.7.

Proof of Theorem 1.13. The first assertion of Theorem 1.13 follows from Corollary 4.6. Suppose that (1.11) holds. The discussion in the introduction shows that (1.11) implies

$$\dim_{\mathbf{F}_q} \left(\bigoplus_{r=0}^{\infty} E_e^{r,n-r} \right) = M_f.$$

Since (1.11) also implies condition (4.11) for $m = n$, it follows from Corollary 4.10 that

$$\dim_{\tilde{\Omega}_0} H^n(\Omega_{C(b)}^\bullet, D) = M_f.$$

5. PROOF OF THEOREM 1.18

Throughout this section, we assume the hypothesis of Theorem 1.18. Put

$$Z^k = \{\omega \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k \mid df^{(\delta)} \wedge \omega = 0\}.$$

We leave it to the reader to check from the definition of the spectral sequence (1.6) that the conclusion of Theorem 1.18 is equivalent to the following assertion.

Proposition 5.1. *Let $k < n$ and let $\omega \in Z^k$ be a homogeneous form such that*

$$(5.2) \quad df^{(\delta')} \wedge \omega = df^{(\delta)} \wedge \xi$$

for some homogeneous form $\xi \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^k$. Then there exist homogeneous forms $\eta_1 \in Z^{k-1}$, $\eta_2 \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k-1}$, such that

$$(5.3) \quad \omega = df^{(\delta')} \wedge \eta_1 + df^{(\delta)} \wedge \eta_2.$$

Before starting the proof, we observe that Kita[9] has, in effect, characterized the elements of Z^k . For notational convenience, we define a 1-form $\Theta \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^1$ by

$$\Theta = f_1 \cdots f_r \sum_{i=1}^r a_i \frac{df_i}{f_i} \quad \left(= f_1 \cdots f_r \frac{df^{(\delta)}}{f^{(\delta)}} \right),$$

and for any l -tuple $1 \leq i_1 < \cdots < i_l \leq r$ we define

$$\Omega_{i_1 \dots i_l} = \frac{df_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{df_{i_l}}{f_{i_l}},$$

a rational l -form with logarithmic poles along the divisor $f_1 \cdots f_r = 0$ in \mathbf{A}^n . Note that $\Theta \wedge \Omega_{i_1 \dots i_l}$ has polynomial coefficients, hence lies in $\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{l+1}$.

Proposition 5.4. *Let $k < n$ and let $\omega \in Z^k$ be homogeneous. Then*

$$(5.5) \quad \omega = \Theta \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \leq i_1 < \cdots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l} \right),$$

for some homogeneous forms $\alpha_{i_1 \dots i_l} \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k-1-l}$.

Proof. The Euler relation and the hypothesis $(p, \delta) = 1$ imply that $f^{(\delta)}$ lies in the ideal generated by the $\partial f^{(\delta)} / \partial x_i$. The equation $df^{(\delta)} \wedge \omega = 0$ then implies, by a standard result ([10, Theorem 16.4], that there exists $\alpha \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k-1}$ such that

$$(5.6) \quad f^{(\delta)} \omega = df^{(\delta)} \wedge \alpha,$$

or, equivalently,

$$(5.7) \quad f_1 \cdots f_r \omega = \Theta \wedge \alpha.$$

This implies that $a_i f_1 \cdots \hat{f}_i \cdots f_r df_i \wedge \alpha \in (f_i)$ for $i = 1, \dots, r$. But $(p, a_i) = 1$, and our hypothesis implies f_j, f_i form a regular sequence for $j \neq i$, hence $df_i \wedge \alpha \in (f_i)$ for $i = 1, \dots, r$. It then follows from [9, Proposition 2.2.3] that

$$(5.8) \quad \alpha = f_1 \cdots f_r \sum_{l=0}^{k-1} \sum_{1 \leq i_1 < \cdots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l},$$

for some homogeneous forms $\alpha_{i_1 \dots i_l} \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k-1-l}$. (Although Kita works over \mathbf{C} , his proof is valid for any field.) Substituting (5.8) into (5.7) gives the proposition.

The following lemma is the main technical tool for the proof of Proposition 5.1.

Lemma 5.9. *Suppose that $\{i_1, \dots, i_l\}$ is a nonempty subset of $\{1, \dots, r\}$ with either $l \geq 2$ or $l = 1$ and $a_{i_1} > 1$ and that $\beta \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^m$ is a homogeneous m -form with $m \leq n - l - 1$ such that*

$$df^{(\delta')} \wedge df_{i_1} \wedge \dots \wedge df_{i_l} \wedge \beta \equiv 0 \pmod{(f_{i_1}, \dots, f_{i_l})}.$$

Then there exist homogeneous $(m-1)$ -forms β_j for $j = 0, 1, \dots, l$ and β'_j for $j = 1, \dots, l$ such that

$$\beta = df^{(\delta')} \wedge \beta_0 + \sum_{j=1}^l df_{i_j} \wedge \beta_j + \sum_{j=1}^l f_{i_j} \beta'_j.$$

Proof. Consider the expansion of $df^{(\delta')} \wedge df_{i_1} \wedge \dots \wedge df_{i_l}$ relative to the basis for $\Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{l+1}$ consisting of the $(l+1)$ -fold exterior products of the dx_i , $i = 1, \dots, n$. By the theorem of [12], it suffices to show that the coefficients in this expansion generate an ideal I of depth $n-l$ in the quotient ring $\mathbf{F}_q[x]/(f_{i_1}, \dots, f_{i_l})$, i. e., the only maximal ideal containing I is the one generated by x_1, \dots, x_n . Suppose there were some other maximal ideal \mathbf{m} containing I . Then \mathbf{m} would correspond to a point in \mathbf{A}^n , other than the origin, which is a common zero of f_{i_1}, \dots, f_{i_l} and at which there is a linear relation between the differentials $df^{(\delta')}, df_{i_1}, \dots, df_{i_l}$. Since $f_{i_1} = \dots = f_{i_l} = 0$ is a smooth complete intersection in \mathbf{A}^n except at the origin, the differentials $df_{i_1}, \dots, df_{i_l}$ are independent, so $df^{(\delta')}$ must be a linear combination of $df_{i_1}, \dots, df_{i_l}$ at this point. But then the Euler relations for the polynomials $f^{(\delta')}, f_{i_1}, \dots, f_{i_l}$, together with the hypothesis that $(p, \delta') = 1$, imply that $f^{(\delta')}$ also vanishes at this point, contradicting the hypothesis that $f^{(\delta')} = f_{i_1} = \dots = f_{i_l} = 0$ defines a smooth complete intersection in \mathbf{A}^n except at the origin.

Let $\omega \in Z^k$ and suppose that for some s , $1 \leq s \leq r$, and b_s, \dots, b_r , $1 \leq b_i \leq a_i$ for $i = s, \dots, r$, we have

$$(5.10) \quad \omega = f_1^{a_1} \dots f_{s-1}^{a_{s-1}} f_s^{b_s} \dots f_r^{b_r} \sum_{i=1}^r a_i \frac{df_i}{f_i} \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \leq i_1 < \dots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l} \right)$$

for some homogeneous forms $\alpha_{i_1 \dots i_l} \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k-1-l}$.

Lemma 5.11. *If $b_s < a_s$, then there exists $\eta \in Z^{k-1}$ such that*

$$(5.12) \quad \omega - df^{(\delta')} \wedge \eta = f_1^{a_1} \dots f_{s-1}^{a_{s-1}} f_s^{b_s} \dots f_r^{b_r} \sum_{i=1}^r a_i \frac{df_i}{f_i} \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \leq i_1 < \dots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha'_{i_1 \dots i_l} \right),$$

where all $\alpha'_{i_1 \dots i_l}$ are divisible by f_s .

Observe that Proposition 5.4 implies that every $\omega \in Z^k$ can be written in the form (5.10) with $s = 1$ and $b_1 = \dots = b_r = 1$. When the conclusion of Lemma 5.11 holds, we can factor out f_s from each $\alpha'_{i_1 \dots i_l}$ and replace b_s by $b_s + 1$ in (5.12). Induction on b_s and s then allows us to replace b_s, \dots, b_r by a_s, \dots, a_r , respectively, in (5.12), giving the following.

Corollary 5.13. *If $\omega \in Z^k$ with $k < n$, then there exists $\eta \in Z^{k-1}$ such that*

$$\omega - df^{(\delta')} \wedge \eta = df^{(\delta)} \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \leq i_1 < \dots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l} \right)$$

for some homogeneous forms $\alpha_{i_1 \dots i_l} \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k-1-l}$.

Proof of Lemma 5.11. Put

$$\begin{aligned} \tilde{\omega} &= \omega / (f_1^{a_1-1} \dots f_{s-1}^{a_{s-1}-1} f_s^{b_s-1} \dots f_r^{b_r-1}) \\ (5.14) \quad &= \Theta \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \leq i_1 < \dots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l} \right). \end{aligned}$$

We show there exists $\tilde{\eta} \in Z^{k-1}$ such that

$$(5.15) \quad \tilde{\omega} - df^{(\delta')} \wedge \tilde{\eta} = \Theta \wedge \left(\sum_{l=0}^{k-1} \sum_{1 \leq i_1 < \dots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha'_{i_1 \dots i_l} \right),$$

where all $\alpha'_{i_1 \dots i_l}$ are divisible by f_s . Lemma 5.11 follows from (5.15) by multiplication by $f_1^{a_1-1} \dots f_{s-1}^{a_{s-1}-1} f_s^{b_s-1} \dots f_r^{b_r-1}$.

Equation (5.2) implies

$$df^{(\delta')} \wedge \tilde{\omega} = f_s^{a_s-b_s} \dots f_r^{a_r-b_r} \Theta \wedge \xi,$$

so if $b_s < a_s$ we have

$$(5.16) \quad df^{(\delta')} \wedge \tilde{\omega} \equiv 0 \pmod{(f_s)}.$$

We prove the existence of $\tilde{\eta}$ by descending induction on l . Since $\alpha_{i_1 \dots i_l} = 0$ for $l > k-1$, we may assume that for some m , $0 \leq m \leq k-1$, we have $\alpha_{i_1 \dots i_l}$ divisible by f_s for all $l \geq m+1$. We show that we can choose $\tilde{\eta}$ so that the $\alpha'_{i_1 \dots i_l}$ are divisible by f_s for all $l \geq m$.

Fix an m -tuple $1 \leq i_1 < \dots < i_m \leq r$ with $s \notin \{i_1, \dots, i_m\}$, say, $i_t < s < i_{t+1}$. Expand the right-hand side of (5.14) using the definitions of Θ and $\Omega_{i_1 \dots i_l}$. The term containing $df_{i_1} \wedge \dots \wedge df_s \wedge \dots \wedge df_{i_m}$ is

$$(5.17) \quad df_{i_1} \wedge \dots \wedge df_s \wedge \dots \wedge df_{i_m} \wedge f_1 \dots \hat{f}_{i_1} \dots \hat{f}_s \dots \hat{f}_{i_m} \dots f_r \left((-1)^t a_s \alpha_{i_1 \dots i_m} + \sum_{j=1}^t (-1)^{j-1} a_{i_j} \alpha_{i_1 \dots \hat{i}_j \dots s \dots i_m} + \sum_{j=t+1}^m (-1)^j a_{i_j} \alpha_{i_1 \dots s \dots \hat{i}_j \dots i_m} \right).$$

Using the induction hypothesis that $\alpha_{i_1 \dots i_l}$ is divisible by f_s for all $l \geq m+1$, one sees that the term containing any other product $df_{i_1} \wedge \dots \wedge df_{i_l}$ (for $l \geq 0$) on the right-hand side of (5.14) lies in the ideal $(f_s, f_{i_1}, \dots, f_{i_m})$. Equation (5.16) then implies

$$(5.18) \quad df^{(\delta')} \wedge df_{i_1} \wedge \dots \wedge df_s \wedge \dots \wedge df_{i_m} \wedge f_1 \dots \hat{f}_{i_1} \dots \hat{f}_s \dots \hat{f}_{i_m} \dots f_r \left((-1)^t a_s \alpha_{i_1 \dots i_m} + \sum_{j=1}^t (-1)^{j-1} a_{i_j} \alpha_{i_1 \dots \hat{i}_j \dots s \dots i_m} + \sum_{j=t+1}^m (-1)^j a_{i_j} \alpha_{i_1 \dots s \dots \hat{i}_j \dots i_m} \right) \equiv 0 \pmod{(f_s, f_{i_1}, \dots, f_{i_m})}.$$

Since ω is a k -form with $k \leq n-1$, we may assume $m \leq n-2$. The smooth complete intersection hypothesis implies that for $j \notin \{s, i_1, \dots, i_m\}$, $f_j, f_s, f_{i_1}, \dots, f_{i_m}$ form a regular sequence, hence (5.18) implies

$$(5.19) \quad df^{(\delta')} \wedge df_{i_1} \wedge \dots \wedge df_s \wedge \dots \wedge df_{i_m} \wedge \\ \left((-1)^t a_s \alpha_{i_1 \dots i_m} + \sum_{j=1}^t (-1)^{j-1} a_{i_j} \alpha_{i_1 \dots i_j \dots s \dots i_m} + \sum_{j=t+1}^m (-1)^j a_{i_j} \alpha_{i_1 \dots s \dots i_j \dots i_m} \right) \\ \equiv 0 \pmod{(f_s, f_{i_1}, \dots, f_{i_m})}.$$

It now follows by Lemma 5.9 (since $b_s < a_s$ implies $a_s > 1$) that there exist homogeneous forms $\gamma_{i_1 \dots i_m}^{(j)}, \delta_{i_1 \dots i_m}^{(j)}$, such that

$$(5.20) \quad \alpha_{i_1 \dots i_m} = \frac{(-1)^t}{a_s} \left(\sum_{j=1}^t (-1)^j a_{i_j} \alpha_{i_1 \dots i_j \dots s \dots i_m} + \sum_{j=t+1}^m (-1)^{j-1} a_{i_j} \alpha_{i_1 \dots s \dots i_j \dots i_m} \right) + \\ df^{(\delta')} \wedge \gamma_{i_1 \dots i_m}^{(0)} + df_s \wedge \gamma_{i_1 \dots i_m}^{(s)} + f_s \delta_{i_1 \dots i_m}^{(s)} + \sum_{j=1}^m df_{i_j} \wedge \gamma_{i_1 \dots i_m}^{(i_j)} + \sum_{j=1}^m f_{i_j} \delta_{i_1 \dots i_m}^{(i_j)}.$$

Such a formula holds for every m -tuple i_1, \dots, i_m not containing s . Substitute these expressions into

$$(5.21) \quad \Theta \wedge \left(\sum_{1 \leq i_1 < \dots < i_m \leq r} \Omega_{i_1 \dots i_m} \wedge \alpha_{i_1 \dots i_m} \right)$$

and expand. After this substitution, only alphas indexed by m -tuples containing s remain. We leave it to the reader to check that for such an m -tuple, say,

$$1 \leq j_1 < \dots < j_t < s < j_{t+1} < \dots < j_{m-1} \leq r,$$

the contribution to (5.21) is

$$\Theta \wedge \frac{(-1)^t \Theta}{a_s f_1 \dots f_r} \wedge \Omega_{j_1 \dots j_{m-1}} \wedge \alpha_{j_1 \dots s \dots j_{m-1}} = 0.$$

Thus after substitution from (5.20), expression (5.21) simplifies to

$$(5.22) \quad \Theta \wedge \left(\sum_{\substack{1 \leq i_1 < \dots < i_m \leq r \\ s \notin \{i_1, \dots, i_m\}}} \Omega_{i_1 \dots i_m} \wedge \right. \\ \left. \left(df^{(\delta')} \wedge \gamma_{i_1 \dots i_m}^{(0)} + df_s \wedge \gamma_{i_1 \dots i_m}^{(s)} + f_s \delta_{i_1 \dots i_m}^{(s)} + \sum_{j=1}^m f_{i_j} \delta_{i_1 \dots i_m}^{(i_j)} \right) \right).$$

We thus take

$$(5.23) \quad \tilde{\eta} = (-1)^{m+1} \Theta \wedge \left(\sum_{\substack{1 \leq i_1 < \dots < i_m \leq r \\ s \notin \{i_1, \dots, i_m\}}} \Omega_{i_1 \dots i_m} \wedge \gamma_{i_1 \dots i_m}^{(0)} \right) \in Z^{k-1},$$

which (by (5.14) and (5.22)) gives

$$(5.24) \quad \tilde{\omega} - df^{(\delta')} \wedge \tilde{\eta} = \Theta \wedge \left(\sum_{\substack{l=0 \\ l \neq m}}^{k-1} \sum_{1 \leq i_1 < \dots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l} + \sum_{\substack{1 \leq i_1 < \dots < i_m \leq r \\ s \notin \{i_1, \dots, i_m\}}} \Omega_{i_1 \dots i_m} \wedge \left(df_s \wedge \gamma_{i_1 \dots i_m}^{(s)} + f_s \delta_{i_1 \dots i_m}^{(s)} + \sum_{j=1}^m f_{i_j} \delta_{i_1 \dots i_m}^{(i_j)} \right) \right).$$

We rewrite this as

$$(5.25) \quad \tilde{\omega} - df^{(\delta')} \wedge \tilde{\eta} = \Theta \wedge \left(\sum_{\substack{l=0 \\ l \neq m}}^{k-1} \sum_{1 \leq i_1 < \dots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l} + \sum_{\substack{1 \leq i_1 < \dots < i_m \leq r \\ s \notin \{i_1, \dots, i_m\}}} \Omega_{i_1 \dots i_m} \wedge f_s \delta_{i_1 \dots i_m}^{(s)} + \sum_{\substack{1 \leq i_1 < \dots < i_m \leq r \\ s \notin \{i_1, \dots, i_m\}}} \Omega_{i_1 \dots s \dots i_m} \wedge (-1)^{m-t} f_s \gamma_{i_1 \dots i_m}^{(s)} + \sum_{\substack{1 \leq i_1 < \dots < i_m \leq r \\ s \notin \{i_1, \dots, i_m\}}} \sum_{j=1}^m \Omega_{i_1 \dots \hat{i}_j \dots i_m} \wedge (-1)^{m-j} df_{i_j} \wedge \delta_{i_1 \dots i_m}^{(i_j)} \right).$$

For $l \geq m+2$, the coefficient of $\Omega_{i_1 \dots i_l}$ on the right-hand side of (5.25) equals $\alpha_{i_1 \dots i_l}$, which is divisible by f_s by the induction hypothesis. For $l = m+1$, the coefficient of $\Omega_{i_1 \dots i_{m+1}}$ equals $\alpha_{i_1 \dots i_{m+1}}$ if $s \notin \{i_1, \dots, i_{m+1}\}$, while the coefficient of $\Omega_{i_1 \dots s \dots i_m}$ equals $\alpha_{i_1 \dots s \dots i_m} + (-1)^{m-t} f_s \gamma_{i_1 \dots i_m}^{(s)}$. In either case, it is divisible by f_s by the induction hypothesis. For $l = m$, the coefficient of $\Omega_{i_1 \dots i_m}$ equals 0 if $s \in \{i_1, \dots, i_m\}$ and equals $f_s \delta_{i_1 \dots i_m}^{(s)}$ if $s \notin \{i_1, \dots, i_m\}$. Thus the coefficient of $\Omega_{i_1 \dots i_l}$ is divisible by f_s for all $l \geq m$, and by induction the proof of Lemma 5.11 is complete.

Proof of Proposition 5.1. Suppose $\omega \in Z^k$ satisfies (5.2). By Corollary 5.13 we may assume that

$$(5.26) \quad \omega = df^{(\delta)} \wedge \sum_{l=1}^{k-1} \sum_{1 \leq i_1 < \dots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l}.$$

Note that we may start the outer sum at $l = 1$ rather than at $l = 0$. We prove by induction on s that for $1 \leq s \leq r$ we can find $\eta_1 \in Z^{k-1}$, $\eta_2 \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k-1}$, such that

$$(5.27) \quad \omega - df^{(\delta')} \wedge \eta_1 - df^{(\delta)} \wedge \eta_2 = df^{(\delta)} \wedge \sum_{l=1}^{k-1} \sum_{1 \leq i_1 < \dots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha'_{i_1 \dots i_l}$$

with all $\alpha'_{i_1 \dots i_l}$ divisible by $f_1 \dots f_s$. When $s = r$, $\Omega_{i_1 \dots i_l} \wedge \alpha'_{i_1 \dots i_l}$ has polynomial coefficients for all i_1, \dots, i_l , so equation (5.27) establishes Proposition 5.1.

So suppose that for some s , $1 \leq s \leq r$, all $\alpha_{i_1 \dots i_l}$ in (5.26) are divisible by $f_1 \dots f_{s-1}$. In this case, if $i_l < s$, then $\Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l} \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k-1}$, so by replacing ω by $\omega - df^{(\delta)} \wedge \Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l}$, we may assume

$$(5.28) \quad \alpha_{i_1 \dots i_l} = 0 \quad \text{when } i_l < s.$$

We prove we can choose η_1, η_2 so that all $\alpha'_{i_1 \dots i_l}$ are divisible by $f_1 \dots f_s$ by descending induction on l . Since $\alpha_{i_1 \dots i_l} = 0$ for $l > k-1$, we may assume that for

some m , $1 \leq m \leq k-1$, $\alpha_{i_1 \dots i_l}$ is divisible by $f_1 \cdots f_s$ for $l \geq m+1$. We show that we can choose η_1, η_2 so that $\alpha'_{i_1 \dots i_l}$ is divisible by $f_1 \cdots f_s$ for $l \geq m$.

Put

$$(5.29) \quad \begin{aligned} \tilde{\omega} &= \omega / (f_1^{a_1-1} \cdots f_r^{a_r-1}) \\ &= \Theta \wedge \left(\sum_{l=1}^{k-1} \sum_{\substack{1 \leq i_1 < \dots < i_l \leq r \\ s \leq i_l}} \Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l} \right). \end{aligned}$$

We show there exist $\tilde{\eta}_1 \in Z^{k-1}$, $\tilde{\eta}_2 \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k-1}$ such that

$$(5.30) \quad \tilde{\omega} - df^{(\delta')} \wedge \tilde{\eta}_1 - \Theta \wedge \tilde{\eta}_2 = \Theta \wedge \sum_{l=1}^{k-1} \sum_{1 \leq i_1 < \dots < i_l \leq r} \Omega_{i_1 \dots i_l} \wedge \alpha'_{i_1 \dots i_l}$$

with all $\alpha'_{i_1 \dots i_l}$ divisible by $f_1 \cdots f_s$. Equation (5.27) follows from (5.30) by multiplication by $f_1^{a_1-1} \cdots f_r^{a_r-1}$.

Equation (5.2) implies

$$(5.31) \quad df^{(\delta')} \wedge \tilde{\omega} \equiv 0 \pmod{(f_s, f_i)}$$

for all $i \neq s$. Fix an m -tuple $1 \leq i_1 < \dots < i_m \leq r$ with $s \notin \{i_1, \dots, i_m\}$ and $s < i_m$, say, $i_t < s < i_{t+1}$. The term containing $df_{i_1} \wedge \dots \wedge df_s \wedge \dots \wedge df_{i_m}$ in the expansion of the right-hand side of (5.30) is given by (5.17). Using the induction hypothesis that $\alpha_{i_1 \dots i_l}$ is divisible by $f_1 \cdots f_s$ for all $l \geq m+1$ and by $f_1 \cdots f_{s-1}$ for all $l \geq 1$, one sees that the term containing any other product $df_{i_1} \wedge \dots \wedge df_{i_l}$ (for $l \geq 1$) on the right-hand side of (5.30) lies in the ideal $(f_s, \{f_{i_j}\}_{i_j > s}, \{f_{i_j}^2\}_{i_j < s})$. Since $\{i_j \mid i_j > s\} \neq \emptyset$, equation (5.31) implies that

$$(5.32) \quad \begin{aligned} &df^{(\delta')} \wedge df_{i_1} \wedge \dots \wedge df_s \wedge \dots \wedge df_{i_m} \wedge \\ &f_1 \cdots \hat{f}_{i_1} \cdots \hat{f}_s \cdots \hat{f}_{i_m} \cdots f_r \left((-1)^t a_s \alpha_{i_1 \dots i_m} + \sum_{j=1}^t (-1)^{j-1} a_{i_j} \alpha_{i_1 \dots \hat{i}_j \dots s \dots i_m} + \right. \\ &\quad \left. \sum_{j=t+1}^m (-1)^j a_{i_j} \alpha_{i_1 \dots s \dots \hat{i}_j \dots i_m} \right) \equiv 0 \pmod{(f_s, \{f_{i_j}\}_{i_j > s}, \{f_{i_j}^2\}_{i_j < s})}. \end{aligned}$$

As noted earlier, we may assume $m \leq n-2$, which implies that $f_j, f_s, f_{i_1}, \dots, f_{i_m}$ is a regular sequence. We then deduce from (5.32) that

$$(5.33) \quad \begin{aligned} &df^{(\delta')} \wedge df_{i_1} \wedge \dots \wedge df_s \wedge \dots \wedge df_{i_m} \wedge \\ &\left((-1)^t a_s \frac{\alpha_{i_1 \dots i_m}}{f_1 \cdots f_{s-1}} + \sum_{j=1}^t (-1)^{j-1} a_{i_j} \frac{\alpha_{i_1 \dots \hat{i}_j \dots s \dots i_m}}{f_1 \cdots f_{s-1}} + \sum_{j=t+1}^m (-1)^j a_{i_j} \frac{\alpha_{i_1 \dots s \dots \hat{i}_j \dots i_m}}{f_1 \cdots f_{s-1}} \right) \\ &\equiv 0 \pmod{(f_s, f_{i_1}, \dots, f_{i_m})}. \end{aligned}$$

Applying Lemma 5.9 (which is permissible since $m \geq 1$) and then multiplying by $f_1 \cdots f_{s-1}$, we see that there exist homogeneous forms $\gamma_{i_1 \dots i_m}^{(j)}$, $\delta_{i_1 \dots i_m}^{(j)}$, all divisible

by $f_1 \cdots f_{s-1}$, such that

$$(5.34) \quad \alpha_{i_1 \dots i_m} = \frac{(-1)^t}{a_s} \left(\sum_{j=1}^t (-1)^j a_{i_j} \alpha_{i_1 \dots \hat{i}_j \dots i_m} + \sum_{j=t+1}^m (-1)^{j-1} a_{i_j} \alpha_{i_1 \dots s \dots \hat{i}_j \dots i_m} \right) +$$

$$df^{(\delta')} \wedge \gamma_{i_1 \dots i_m}^{(0)} + df_s \wedge \gamma_{i_1 \dots i_m}^{(s)} + f_s \delta_{i_1 \dots i_m}^{(s)} + \sum_{j=1}^m df_{i_j} \wedge \gamma_{i_1 \dots i_m}^{(i_j)} + \sum_{j=1}^m f_{i_j} \delta_{i_1 \dots i_m}^{(i_j)}.$$

Such a formula holds for every m -tuple i_1, \dots, i_m not containing s with $s < i_m$. Substitute these expressions into

$$(5.35) \quad \Theta \wedge \left(\sum_{\substack{1 \leq i_1 < \dots < i_m \leq r \\ s \leq i_m}} \Omega_{i_1 \dots i_m} \wedge \alpha_{i_1 \dots i_m} \right).$$

After this substitution, only alphas indexed by m -tuples containing s remain. Consider such an m -tuple, say,

$$1 \leq j_1 < \dots < j_t < s < j_{t+1} < \dots < j_{m-1} \leq r.$$

If $t < m-1$, the reader may check that the contribution to (5.35) is

$$\Theta \wedge \frac{(-1)^t \Theta}{a_s f_1 \cdots f_r} \wedge \Omega_{j_1 \dots j_{m-1}} \wedge \alpha_{j_1 \dots s \dots j_{m-1}} = 0.$$

In the case $t = m-1$ (i. e., $j_i < s$ for all i), the reader may check that the contribution is

$$\begin{aligned} \Theta \wedge \frac{(-1)^{m-1}}{a_s} \sum_{i=s}^r a_i \Omega_{ij_1 \dots j_{m-1}} \wedge \alpha_{j_1 \dots j_{m-1} s} \\ = \Theta \wedge \frac{(-1)^m}{a_s} \sum_{i=1}^{s-1} a_i \Omega_{ij_1 \dots j_{m-1}} \wedge \alpha_{j_1 \dots j_{m-1} s}, \end{aligned}$$

since $\sum_{i=1}^r a_i \Omega_{ij_1 \dots j_{m-1}} = \Theta \wedge \Omega_{j_1 \dots j_{m-1}}$. Put

$$\tilde{\eta}_2 = \frac{(-1)^m}{a_s} \sum_{i=1}^{s-1} a_i \Omega_{ij_1 \dots j_{m-1}} \wedge \alpha_{j_1 \dots j_{m-1} s}.$$

For $i = 1, \dots, s-1$, we have $\{i, j_1, \dots, j_{m-1}\} \subseteq \{1, \dots, s-1\}$, and since $\alpha_{j_1 \dots j_{m-1} s}$ is divisible by $f_1 \cdots f_{s-1}$, we conclude that $\tilde{\eta}_2 \in \Omega_{\mathbf{F}_q[x]/\mathbf{F}_q}^{k-1}$.

Thus after substitution from (5.34), expression (5.35) simplifies to

$$(5.36) \quad \Theta \wedge \left(\tilde{\eta}_2 + \sum_{\substack{1 \leq i_1 < \dots < i_m \leq r \\ s \notin \{i_1, \dots, i_m\} \\ s < i_m}} \Omega_{i_1 \dots i_m} \wedge \right.$$

$$\left. \left(df^{(\delta')} \wedge \gamma_{i_1 \dots i_m}^{(0)} + df_s \wedge \gamma_{i_1 \dots i_m}^{(s)} + f_s \delta_{i_1 \dots i_m}^{(s)} + \sum_{j=1}^m f_{i_j} \delta_{i_1 \dots i_m}^{(i_j)} \right) \right).$$

We now take

$$(5.37) \quad \tilde{\eta}_1 = (-1)^{m+1} \Theta \wedge \left(\sum_{\substack{1 \leq i_1 < \dots < i_m \leq r \\ s \notin \{i_1, \dots, i_m\} \\ s < i_m}} \Omega_{i_1 \dots i_m} \wedge \gamma_{i_1 \dots i_m}^{(0)} \right) \in Z^{k-1}$$

and conclude (see (5.29))

$$(5.38) \quad \tilde{\omega} - df^{(\delta')} \wedge \tilde{\eta}_1 - \Theta \wedge \tilde{\eta}_2 = \Theta \wedge \left(\sum_{\substack{l=0 \\ l \neq m}}^{k-1} \sum_{\substack{1 \leq i_1 < \dots < i_l \leq r \\ s \leq i_l}} \Omega_{i_1 \dots i_l} \wedge \alpha_{i_1 \dots i_l} + \right. \\ \left. \sum_{\substack{1 \leq i_1 < \dots < i_m \leq r \\ s \notin \{i_1, \dots, i_m\} \\ s < i_m}} \Omega_{i_1 \dots i_m} \wedge \left(df_s \wedge \gamma_{i_1 \dots i_m}^{(s)} + f_s \delta_{i_1 \dots i_m}^{(s)} + \sum_{j=1}^m f_{i_j} \delta_{i_1 \dots i_m}^{(i_j)} \right) \right).$$

Rewriting the right-hand side of (5.38) as in (5.25) and arguing as we did in the proof of Lemma 5.11 shows that the coefficient of $\Omega_{i_1 \dots i_l}$ is divisible by $f_1 \cdots f_s$ for $l \geq m$. This completes the proof of Proposition 5.1.

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DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OKLAHOMA 74078
E-mail address: `adolphs@math.okstate.edu`

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MINNESOTA 55455
E-mail address: `sperber@math.umn.edu`